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The analysis and modelling of dilatational terms in compressible turbulence

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It is shown that the dilatational terms that need to be modelled in compressible turbulence include not only the pressure–dilatation term but also another term – the compressible dissipation. The nature of the compressible velocity field, which generates these dilatational terms, is explored by asymptotic analysis of the compressible Navier–Stokes equations in the case of homogeneous turbulence. The lowest-order equations for the compressible field are solved and explicit expressions for some of the associated one-point moments are obtained. For low Mach numbers, the compressible mode has a fast timescale relative to the incompressible mode. Therefore, it is proposed that, in moderate Mach number homogeneous turbulence, the compressible component of the turbulence is in quasi-equilibrium with respect to the incompressible turbulence. A non-dimensional parameter which characterizes this equilibrium structure of the compressible mode is identified. Direct numerical simulations (DNS) of isotropic, compressible turbulence are performed, and their results are found to be in agreement with the theoretical analysis. A model for the compressible dissipation is proposed; the model is based on the asymptotic analysis and the direct numerical simulations. This model is calibrated with reference to the DNS results regarding the influence of compressibility on the decay rate of isotropic turbulence. An application of the proposed model to the compressible mixing layer has shown that the model is able to predict the dramatically reduced growth rate of the compressible mixing layer.

1. Introduction

When the Mach number of a turbulent flow increases, the fluctuations in the thermodynamic variables – density, temperature and pressure – become progressively more important. The velocity field can no longer be assumed to be solenoidal when the flow Mach number is significant. Also, the turbulent flow radiates sound into the ambient fluid. The differences in the nature of the various fluctuating fields in a compressible medium have been illustrated by Kovasznay (1953) through the decomposition of the turbulence into three components, namely, the vorticity, acoustic and entropy modes. Turbulence modelling for compressible flows has to account for the additional correlations involving both the fluctuating thermodynamic quantities and the fluctuating dilatation. In low-speed flows too, significant fluctuations in density and dilatation can occur in various situations, such as, the mixing of fluids with different densities, turbulent combustion, and turbulent boundary layers with strongly heated walls. This paper is concerned with only high-speed flows. The role of thermodynamic and dilatational fluctuations in low-speed flows is probably different from that in high-speed flows; for example, a supersonic shear layer at Mach 3 shows significant reduction in the growth rate of its width relative to the incompressible shear layer; however, a low-speed, variable-density shear layer having the same density difference as the Mach 3 shear layer exhibits a relatively mild change in growth rate with respect to its constant-density counterpart.

Among the various additional correlations introduced into the problem owing to compressibility, only the class of correlations involving the fluctuating dilatation is considered here. The need for modelling the pressure–dilatation is generally accepted; we show, however, that there is another dilatational correlation – the compressible dissipation – which also merits attention.

According to Morkovin's hypothesis (Morkovin 1964; Bradshaw 1977), direct compressible effects on the turbulence may be ignored when the ratio of the root-mean-square (r.m.s.) density fluctuations to the mean density is small. Consequently (Bradshaw 1977), variable mean density extensions of incompressible turbulence models are expected to give good results in turbulent boundary layers with the free-stream Mach number $M < 5$, and in compressible jets with $M < 1.5$. Apart from the intensity of the density fluctuations, there is another indicator of the intrinsic compressibility of high-speed turbulence: the turbulent Mach number $M_t = q/\bar{c}$, where q^2 is twice the turbulent kinetic energy, and $\bar{c}$ is the local mean speed of sound. The turbulent Mach number may be a more direct gauge of compressibility effects than the ratio of density fluctuations; for example, the latter quantity is large in the shear layer between two low-speed streams with different densities, however, the turbulent Mach number is small, and the change in growth rate is also correspondingly small. In the present paper, using asymptotic theory and direct numerical simulations, we show that the compressible dissipation is naturally related to the turbulent Mach number.

Turbulent fluctuations, which are contained in a bounded domain, radiate sound into the surrounding region. The nature of the sound field at large distances from the turbulence is one of the major concerns of aero-acoustics. In order to predict the radiated sound, it is necessary to characterize properly the coupled density and velocity fields inside the flow domain, which of course is a primary concern of compressible turbulence modelling. For small Mach number, the radiated sound field is dominated by frequencies corresponding to the characteristic frequencies of the energy containing eddies (see Crighton 1975), and has been shown by Lighthill (1952) to carry a very small fraction M_t^5 of the energy required to sustain the turbulence in the bounded domain. We note that the present work does not treat the problem of sound propagation far away from the turbulence, but rather focuses on the changes in the stochastic quantities inside a turbulent flow, which are induced by the compressibility of the medium.

The presence of shock waves is an important feature that distinguishes the high-speed flows from the low-speed ones. It is known that the interaction of a shock wave with a turbulent boundary layer leads to a significant increase in turbulence intensity and shear stress across the shock (Sekundoz 1974; Mateer, Brosh & Viegas 1976; Delery 1981). Some of the basic mechanisms underlying the shock wave/turbulence interaction have been investigated through the numerical solutions (Zang, Hussaini & Bushnell 1984) of the Euler equations. Such compressibility effects may preclude successful extension of incompressible turbulence models to include compressibility solely through the variability of the mean density.

The paper is organized as follows. In §2 the dilatational terms that need to be modelled in the Reynolds stress transport equation are formally obtained. In §3 the dilatational terms are analysed by an asymptotic theory; one of the main results of this section is the identification of a non-dimensional parameter F which is asymptotically equal to unity for low Mach number, compressible, homogeneous turbulence. In §4 results of three-dimensional direct numerical simulations (DNS) of moderate Mach number, isotropic turbulence are presented and shown to be in good agreement with the theoretical findings of §3. In §5 a model for the compressible dissipation, which is based on the asymptotic analysis and the DNS, is proposed; the model is calibrated with reference to the DNS results on the decay rate of compressible, isotropic turbulence; and an application by Sarkar & Balakrishnan (1990) of the new model to the compressible shear layer is briefly considered. Conclusions are presented in §6.

2. Dilatational terms in the turbulence transport equations

In this section, we identify the correlations involving the fluctuating dilatation that need to be modelled in the Reynolds stress transport equations. It is shown that in addition to the well-known pressure–dilatation, an additional term, the *compressible dissipation*, needs to be modelled.

The compressible Navier–Stokes equations, along with an equation of state, govern the behaviour of the density ρ, the velocity u_i, the temperature T and the pressure p in a high-speed, compressible flow. When the compressible flow is turbulent, an averaged form of the compressible Navier–Stokes equations is usually considered, wherein the instantaneous variables are decomposed into a mean and a fluctuating part, and the governing equations are averaged in order to yield equations for the mean variables. Usually Favre averages (density-weighted averages) are used for the velocity and temperature, while conventional Reynolds averages are used for the pressure and density; primarily, because such a combination leads to a simpler representation of the temporal derivatives and the convective terms in the averaged equations. We employ the above-mentioned approach too, and decompose the field variables as follows,

$$u_i = \tilde{u}_i + u_i'', \quad \rho = \bar{\rho} + \rho',$$
$$T = \tilde{T} + T'', \quad p = \bar{p} + p'.$$

The overbar denotes the conventional Reynolds average and the prime denotes fluctuations with respect to the Reynolds average, while the tilde denotes the Favre average and the double prime denotes fluctuations with respect to the Favre average. The Favre average $\tilde{\phi}$ of a field variable ϕ is a density-weighted Reynolds average;

$$\tilde{\phi} = \overline{\rho\phi}/\bar{\rho}.$$

We consider a second-order turbulence closure where in addition to the mean equations, transport equations are included for the Favre-averaged Reynolds stress $\widetilde{u_i'' u_j''}$ and the turbulence dissipation rate ϵ. The exact transport equation for $\widetilde{u_i'' u_j''}$ is,

$$\partial_t(\bar{\rho}\widetilde{u_i'' u_j''}) + (\bar{\rho}\tilde{u}_k \widetilde{u_i'' u_j''})_{,k} = P_{ij} - T_{ijk,k} + \Pi_{ij} - \bar{\rho}\epsilon_{ij} + \tfrac{2}{3}\overline{p' u'_{k,k}}\,\delta_{ij} - \overline{u_i''}\,\bar{p}_{,j} - \overline{u_j''}\,\bar{p}_{,i} + \overline{u_i''}\,\bar{\sigma}_{jk,k} + \overline{u_j''}\,\bar{\sigma}_{ik,k}. \quad (1)$$

where

$$P_{ij} = -\bar{\rho}(\widetilde{u_i'' u_k''}\,\tilde{u}_{j,k} + \widetilde{u_j'' u_k''}\,\tilde{u}_{i,k}),$$
$$T_{ijk} = \bar{\rho}\widetilde{u_i'' u_j'' u_k''} + (\overline{p' u_i'}\,\delta_{jk} + \overline{p' u_j'}\,\delta_{ik}) - (\overline{u_i' \sigma_{jk}'} + \overline{u_j' \sigma_{ik}'}),$$
$$\Pi_{ij} = \overline{p' u'_{i,j}} + \overline{p' u'_{j,i}} - \tfrac{2}{3}\overline{p' u'_{k,k}}\,\delta_{ij},$$
$$\bar{\rho}\epsilon_{ij} = \overline{\sigma_{ik}' u'_{j,k}} + \overline{\sigma_{jk}' u'_{i,k}}.$$

In (1), P_{ij} denotes the production, T_{ijk} denotes the diffusive transport, Π_{ij} denotes the deviatoric part of the pressure–strain correlation, and ϵ_{ij} denotes the turbulent dissipation rate tensor. The anisotropic part of ϵ_{ij} is usually absorbed into the model for the pressure–strain correlation, while the trace of ϵ_{ij} is written in terms of the turbulent rate ϵ as given below,

$$\bar{\rho}\epsilon = \overline{\sigma'_{ij} u'_{i,j}}.$$

We note that the conventional Reynolds average of the Favre fluctuation u''_i is non-zero; in fact, $\overline{u''_i}$ is related to the turbulent mass flux $\overline{\rho' u'_i}$ by the expression

$$\overline{u''_i} = -\frac{\overline{\rho' u'_i}}{\bar{\rho}}.$$

At first glance, it appears that the only term in (1) which contains the fluctuating dilatation $d' = u'_{k,k}$ is the pressure–dilatation $\overline{p'd'}$. However, we show below that in a compressible flow there is another term containing d', which has its origins in the turbulent dissipation rate ϵ. The viscous stress σ_{ij} in a compressible flow is given by

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) - \tfrac{2}{3}\mu d \delta_{ij},$$

where we have assumed that the bulk viscosity is zero. Assuming constant viscosity, the following expression for the turbulent dissipation rate is obtained:

$$\begin{aligned}\bar{\rho}\epsilon &= \overline{\sigma'_{ij} u'_{i,j}} \\ &= \mu(2\overline{s'_{kl} s'_{kl}} - \tfrac{2}{3}\overline{d'^2}),\end{aligned}$$

where the fluctuating strain rate $s'_{kl} = \tfrac{1}{2}(u'_{k,l} + u'_{l,k})$. Even if the viscosity is not assumed constant, standard order of magnitude estimates lead to the following expression for the turbulent dissipation rate, which is asymptotically exact for high turbulence Reynolds number:

$$\bar{\rho}\epsilon = \bar{\mu}(2\overline{s'_{kl} s'_{kl}} - \tfrac{2}{3}\overline{d'^2}). \tag{2}$$

Let us denote the fluctuating vorticity tensor by $w'_{kl} = \tfrac{1}{2}(u'_{k,l} - u'_{l,k})$ and the fluctuating vorticity by ω'_i. On substituting the relationship

$$\overline{s'_{kl} s'_{kl}} = \overline{w'_{kl} w'_{kl}} + \overline{u'_{k,l} u'_{l,k}}$$

into (2), we obtain

$$\bar{\rho}\epsilon = \bar{\mu}(2\overline{w'_{kl} w'_{kl}} + 2\overline{u'_{k,l} u'_{l,k}} - \tfrac{2}{3}\overline{d'^2}). \tag{3}$$

The scalar $\overline{u'_{k,l} u'_{l,k}}$ satisfies the following equation (which may be verified by inspection):

$$\overline{u'_{k,l} u'_{l,k}} = (\overline{u'_k u'_l})_{,kl} - 2(\overline{u'_{k,k} u'_l})_{,l} + \overline{u'_{k,k} u'_{l,l}}. \tag{4}$$

For homogeneous turbulence, (4) becomes the rather simple expression

$$\begin{aligned}\overline{u'_{k,l} u'_{l,k}} &= \overline{u'_{k,k} u'_{l,l}} \\ &= \overline{d'^2}.\end{aligned} \tag{5}$$

For inhomogeneous turbulence, using standard order of magnitude estimates, (5) may be shown to be asymptotically correct for high turbulence Reynolds number. On substituting (5) into (3), we obtain

$$\begin{aligned}\bar{\rho}\epsilon &= \bar{\mu}(2\overline{w'_{kl} w'_{kl}} + \tfrac{4}{3}\overline{d'^2}) \\ &= \bar{\mu}(\overline{\omega'_i \omega'_i} + \tfrac{4}{3}\overline{d'^2}).\end{aligned} \tag{6}$$

Thus, we have shown that for compressible turbulence the dissipation rate may be decomposed into

$$\bar{\rho}\epsilon = \bar{\rho}\epsilon_{\mathrm{s}} + \bar{\rho}\epsilon_{\mathrm{c}}, \tag{7}$$

where

$$\bar{\rho}\epsilon_{\mathrm{s}} = \bar{\mu}\overline{\omega'_i \omega'_i}, \tag{8}$$

and

$$\bar{\rho}\epsilon_{\mathrm{c}} = \tfrac{4}{3}\bar{\mu}\overline{d'^2}. \tag{9}$$

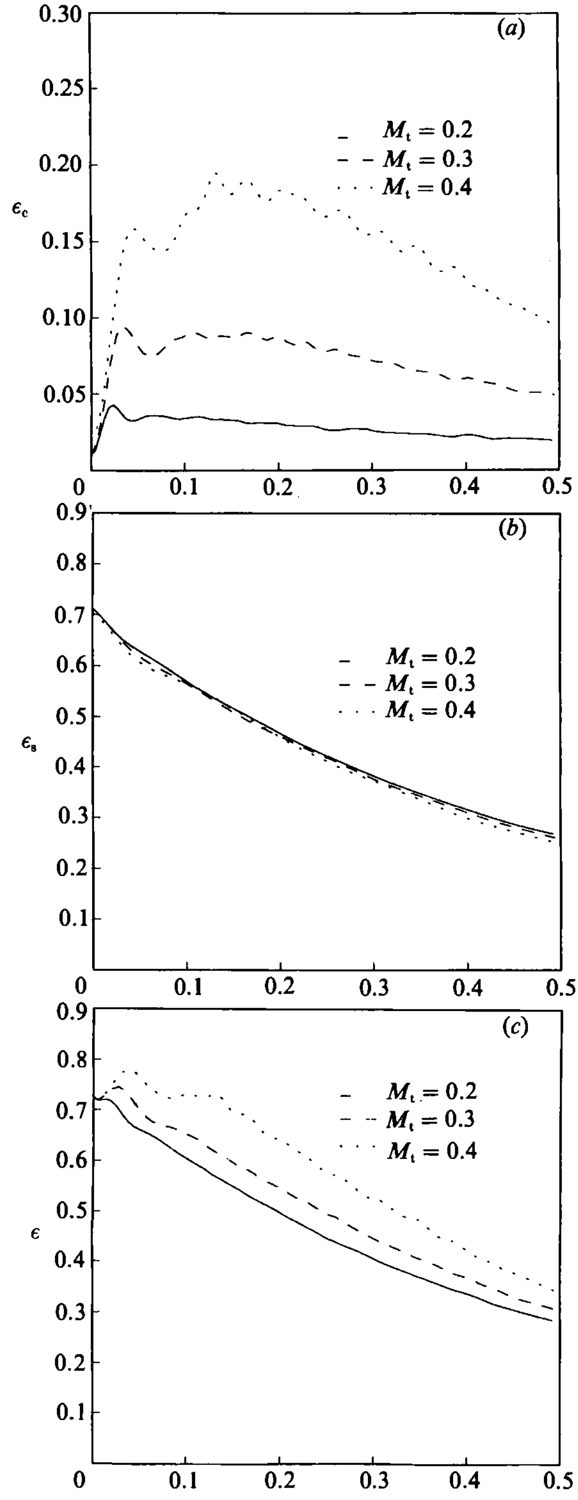

FIGURE 1. For various DNS cases: (*a*) Behaviour of the compressible dissipation. (*b*) Behaviour of the solenoidal dissipation. (*c*) Behaviour of the net dissipation.

Equation (7) is asymptotically exact for turbulence with high Reynolds number (which is of practical interest) and is exact for constant viscosity, homogeneous turbulence (which corresponds to the direct simulations discussed later).

We call the component ϵ_s, which is associated with the vortical component of the velocity field, the *solenoidal dissipation*, while the component ϵ_c, which is associated with the dilatational component of the velocity field, is called the *compressible dissipation*. The asymptotic analysis of §3 leads to the result (63), which implies that, in moderate Mach-number turbulence, only ϵ_c is substantially affected by changes of compressibility indicators such as the turbulent Mach number, while the fluctuating vorticity field and thereby ϵ_s is relatively unaffected by such changes. The direct numerical simulation results of figure 1 also show that moderate compressibility affects ϵ_c and not ϵ_s.

Zeman (1990) has also independently used a similar decomposition of the dissipation rate into a solenoidal and a compressible part. Zeman considers the presence of eddy shocklets which are assumed to augment only the compressible dissipation, bypassing the solenoidal energy cascade. We, on the other hand, identify the compressible and solenoidal parts of the turbulent dissipation and obtain the scaling of the compressible dissipation by asymptotic analysis of compressible turbulence, and validate the decomposition and scaling with direct numerical simulations.

For polyatomic gases, the bulk viscosity may be comparable in magnitude to the shear viscosity μ and lead to an extra dissipation which has a functional form similar to that of ϵ_c. The additional turbulent dissipation due to the non-negligible bulk viscosity can be easily modelled in the same way as the compressible dissipation ϵ_c is modelled in §5.

3. Low-Mach-number asymptotics

The dimensional variables which are denoted by superscript * are non-dimensionalized as follows:

$$l = \frac{l^*}{l_r^*}, \quad u = \frac{u^*}{u_r^*}, \quad t = \frac{t^* u_r^*}{l_r^*}, \quad \rho = \frac{\rho^*}{\rho_r^*}, \quad p = \frac{p^*}{p_r^*}, \quad T = \frac{T^*}{T_r^*},$$

where l_r^* and u_r^* denote a characteristic turbulence integral lengthscale and a turbulence velocity scale, respectively; ρ_r^*, p_r^*, and $T_r^* = p_r^*/R\rho_r^*$ denote reference values for the density, thermodynamic pressure, and static temperature, respectively; and R denotes the gas constant. Reference values for the kinematic viscosity and the thermal diffusivity are respectively denoted by ν_r^* and α_r^*. After using the above non-dimensionalization, the compressible Navier–Stokes equations take the form,

$$\partial_t \rho + u_i \rho_{,i} = -\rho u_{i,i}, \tag{10}$$

$$\rho \partial_t u_i + \rho u_j u_{i,j} = -\frac{1}{\gamma M_r^2} p_{,i} + \frac{1}{Re_r} \sigma_{ij,j}, \tag{11}$$

$$\partial_t p + u_j p_{,j} = -\gamma p u_{i,i} - \frac{\gamma}{Pr_r Re_r} q_{i,i} + \gamma(\gamma-1) \frac{M_r^2}{Re_r} \sigma_{ij} u_{i,j}. \tag{12}$$

The variables σ_{ij} and q_i denote the viscous stress tensor and the heat flux, respectively. The non-dimensional parameters appearing in (10)–(12) are the Mach number $M_r = u_r^*/(\gamma p_r^*/\rho_r^*)^{\frac{1}{2}}$, the Reynolds number $Re_r = u_r^* l_r^*/\nu_r^*$ and the Prandtl number $Pr_r = \nu_r^*/\alpha_r^*$.

We now consider homogeneous, compressible turbulence and adopt the approach of Erlebacher *et al.* (1991), in which the velocity is split into an incompressible, solenoidal velocity u_i^{I} and a compressible velocity u_i^{C}; and the pressure change with respect to the reference pressure is correspondingly split into an incompressible pressure p^{I} and a compressible pressure p^{C}, as follows:

$$u_i = u_i^{\mathrm{I}} + u_i^{\mathrm{C}}, \quad p = 1 + \gamma M_{\mathrm{r}}^2 p^{\mathrm{I}} + p^{\mathrm{C}}. \tag{13}$$

The variables p^{I} and u_i^{I} satisfy the incompressible problem, i.e.

$$\partial_t u_i^{\mathrm{I}} + u_j^{\mathrm{I}} u_{i,j}^{\mathrm{I}} = -p_{,i}^{\mathrm{I}} + \frac{1}{Re_{\mathrm{r}}} u_{i,jj}^{\mathrm{I}}, \tag{14}$$

$$u_{i,i}^{\mathrm{I}} = 0. \tag{15}$$

The following set of equations for u_i^{C} and p^{C}, has been derived and discussed by Erlebacher *et al.* (1991).

$$\partial_t u_i^{\mathrm{C}} + u_j^{\mathrm{I}} u_{i,j}^{\mathrm{C}} + u_j^{\mathrm{C}} u_{i,j}^{\mathrm{C}} + u_j^{\mathrm{C}} u_{i,j}^{\mathrm{I}} + \frac{p_{,i}^{\mathrm{C}}}{\gamma M_{\mathrm{r}}^2} = 0, \tag{16}$$

$$\partial_t p^{\mathrm{C}} + u_i^{\mathrm{I}} p_{,i}^{\mathrm{C}} + u_i^{\mathrm{C}} p_{,i}^{\mathrm{C}} + \gamma M_{\mathrm{r}}^2 u_i^{\mathrm{C}} p_{,i}^{\mathrm{I}} + \gamma(1 + \gamma M_{\mathrm{r}}^2 p^{\mathrm{I}} + p^{\mathrm{C}}) u_{i,i}^{\mathrm{I}} = -\gamma M_{\mathrm{r}}^2 (\partial_t p^{\mathrm{I}} + u_i^{\mathrm{I}} p_{,i}^{\mathrm{I}}). \tag{17}$$

Equations (16)–(17) have been obtained by subtracting the incompressible problem (14)–(15) from the Navier–Stokes equations and dropping the viscous and heat conduction terms.

The above problem can be investigated by an asymptotic analysis, wherein the Mach number M_{r} is considered to be a small parameter, and the pressure is expanded in a power series with respect to M_{r}. The leading-order term in the asymptotic series for the compressible pressure p^{C} is written as

$$p^{\mathrm{C}} = \delta P, \tag{18}$$

where $\delta = O(M_{\mathrm{r}}^n)$ and $P = O(1)$.

When the Mach number M_{t} characterizing the large-scale turbulence eddies is small, the incompressible and compressible (acoustic) timescales are disparate. For a large eddy of characteristic length l and characteristic velocity u, the turbulence timescale is $t_{\mathrm{T}} = O(l/u)$ while the acoustic timescale is $t_{\mathrm{C}} = O(l/c)$. Here c is the characteristic sound speed. Since, $t_{\mathrm{C}}/t_{\mathrm{T}} = O(u/c) = O(M_{\mathrm{t}})$, this acoustic timescale is small relative to the incompressible turbulence timescale when M_t is small.

We now consider the compressible problem on the small acoustic timescale t_{C}, which allows the neglect of the convective and viscous terms in (16)–(17). The lowest-order problem for the compressible fluctuations is the following set of equations:

$$\partial_t P' + \frac{\gamma}{\delta} u_{i,i}^{\mathrm{C}'} = 0, \tag{19}$$

$$\partial_t u_i^{\mathrm{C}'} + \frac{\delta}{\gamma M_{\mathrm{r}}^2} P'_{,i} = 0. \tag{20}$$

We refer to (19)–(20) as the acoustic truncation of the equations governing the compressible component. The zero subscript is used to denote the initial value $\phi(x_i, 0)$ of a variable $\phi(x_i, t)$; for example,

$$u_i^{\mathrm{C}'}(x_i, 0) = (u_i^{\mathrm{C}'})_0(x_i), \quad P'(x_i, 0) = (P')_0(x_i). \tag{21}$$

The initial compressible velocity field $(u_i^{\mathrm{C}'})_0$ satisfies the conditions

$$\nabla \times (u_i^{\mathrm{C}'})_0 = 0, \quad \nabla \cdot (u_i^{\mathrm{C}'})_0 = (d')_0.$$

Thus, a Helmholtz decomposition is performed on the initial velocity field, and the initial value for the compressible velocity field is chosen to be irrotational and dilatational, while the initial value for the incompressible velocity field is chosen to be rotational and solenoidal. We note that for the homogeneous flows considered here, $\int \boldsymbol{u}\cdot\boldsymbol{n}$ is zero on the boundary, and the Helmholtz decomposition is unique.

Let us denote the vorticity $\nabla\times u_i^{C'}$ associated with the compressible velocity by $\omega_i^{C'}$, and the dilatation $\nabla\cdot u_i^{C'}$ associated with the compressible velocity by d'. On taking the curl of (20), and making use of the initial condition $(\omega_i^{C'})_0 = 0$, it follows that $\omega_i^{C'}(x_i, t) = 0$. Thus the compressible velocity remains irrotational under the acoustic truncation of the governing equations. We emphasize that the actual u_i^C at a given time is not the irrotational part of the velocity field obtained by the Helmholtz decomposition of the full velocity field at that time into rotational and irrotational parts. Rather, u_i^C satisfies a set of evolution equations, and the full compressible problem (without the acoustic truncation) allows u_i^C to acquire a small amount of vorticity (estimated later in this section) as it evolves in time from its initial irrotational state.

After some manipulation, (19) and (20) yield

$$\partial_{tt} P' - \frac{1}{M_r^2} P'_{,ii} = 0, \tag{22}$$

$$\partial_{tt} d' - \frac{1}{M_r^2} d'_{,ii} = 0. \tag{23}$$

The above wave equations for the pressure and the dilatation are coupled through the initial conditions; the initial conditions for (22) are

$$P'(x_i, 0) = (P')_0, \quad \partial_t P'(x_i, 0) = -\frac{\gamma}{\delta}(d')_0 \tag{24}$$

while those for (23) are,

$$d'(x_i, 0) = (d')_0, \quad \partial_t d'(x_i, 0) = -\frac{\delta}{\gamma M_r^2}(P'_{,ii})_0. \tag{25}$$

The explicit appearance of M_r in (22) and (23) is removed by rescaling time through the transformation

$$\tau = \frac{t}{M_r}. \tag{26}$$

After using (26) to rescale time, the equations for P' and d' take the form

$$\partial_{\tau\tau} P' - P'_{,ii} = 0, \tag{27}$$

$$\partial_{\tau\tau} d' - d'_{,ii} = 0. \tag{28}$$

The initial conditions are

$$P'(x_i, 0) = (P')_0, \quad \partial_\tau P'(x_i, 0) = -\frac{1}{M_r^*}(d')_0, \tag{29}$$

and

$$d'(x_i, 0) = (d')_0, \quad \partial_\tau d'(x_i, 0) = -M_r^*(P'_{,ii})_0, \tag{30}$$

where the quantity M_r^* is defined by the expression,

$$M_r^* = \frac{\delta}{\gamma M_r}. \tag{31}$$

We define the turbulent Mach number M_t as

$$M_t = q/\bar{c}, \tag{32}$$

where $$q^2 = \widetilde{u_i'' u_i''},$$

and $\bar{c}$ is the mean speed of sound. The quantity M_r is a constant reference Mach number which appears when the Navier–Stokes equations are non-dimensionalized, while M_t is a local time-dependent and space-dependent Mach number characterizing the turbulent fluctuations. In an initial-value problem of isotropic turbulence, M_r is chosen to be of the same order as the initial value of M_t. We can rewrite (31) in terms of the turbulent Mach number M_t as follows,

$$M_r^* = \frac{\delta q}{\gamma M_t} \frac{c_r}{\bar{c}}, \tag{33}$$

where $c_r = c_r^*/u_r^*$ is the non-dimensional, reference speed of sound.

The system of equations (27) and (28) for the pressure P' and the divergence d' are now solved using Fourier transforms. The Fourier transforms $\hat{P}(k_i)$ and $\hat{d}(k_i)$ are defined below,

$$\hat{P}(k_i, \tau) = \frac{1}{(2\pi)^3} \int \exp(\mathrm{i} k_i x_i) P'(x_i, \tau)\, \mathrm{d}x_i,$$

$$\hat{d}(k_i, \tau) = \frac{1}{(2\pi)^3} \int \exp(\mathrm{i} k_i x_i) d'(x_i, \tau)\, \mathrm{d}x_i,$$

where k_i denotes the wavenumber vector. In Fourier transform space, (27) and (28) take the form

$$\partial_{\tau\tau} \hat{P} + k^2 \hat{P} = 0, \tag{34}$$

$$\partial_{\tau\tau} \hat{d} + k^2 \hat{d} = 0, \tag{35}$$

while the initial conditions become

$$\hat{P}(k_i, 0) = \hat{P}_0, \quad \partial_\tau \hat{P}(k_i, 0) = -\frac{\hat{d}_0}{M_r^*}, \tag{36}$$

and $$\hat{d}(k_i, 0) = \hat{d}_0, \quad \partial_\tau \hat{d}(k_i, 0) = M_r^*\, k^2 \hat{P}_0. \tag{37}$$

It is a simple matter to obtain the solution of (34) and (35) which satisfies the initial condition (36) and (37). The solution is

$$\hat{P}(k_i, \tau) = \hat{P}_0 \cos k\tau - \frac{\hat{d}_0}{k M_r^*} \sin k\tau, \tag{38}$$

$$\hat{d}(k_i, \tau) = \hat{d}_0 \cos k\tau + \hat{P}_0\, k M_r^* \sin k\tau. \tag{39}$$

Equations (38) and (39) represent solutions (in Fourier space) for the compressible pressure fluctuation P' and the fluctuating dilatation d'; these solutions were obtained by analysis of the acoustic truncation of the compressible problem. Recalling that the transformed time coordinate τ is related to the time t through (26), it is clear that the evolution of the compressible pressure and the dilatation from their initial values occurs in a non-dimensional time $t = O(M_r)$. Thus for small M_r, the compressible part of the problem is associated with a fast timescale.

It is now possible to obtain solutions for the pressure variance $\overline{P'^2}(\tau)$, the pressure–dilatation $\overline{P'd'}(\tau)$, and the dilatational variance $\overline{d'^2}(\tau)$. By definition, the pressure variance is related to the Fourier transform of the pressure as follows:

$$\overline{P'^2}(\tau) = \frac{(2\pi)^3}{V} \int_{-\infty}^{\infty} |\hat{P}(k_i, \tau)|^2\, \mathrm{d}k_i, \tag{40}$$

where V is the (sufficiently large) averaging volume, and $|\hat{P}(k_i,\tau)|$ denotes the modulus of the complex variable $\hat{P}(k_i,\tau)$. Substituting (38) for $\hat{P}(k_i,\tau)$ into (40) leads to the following expression for the pressure variance,

$$\overline{P'^2}(\tau) = \int_0^\infty E_{P,0}(k)\cos^2 k\tau\,\mathrm{d}k + \frac{1}{M_\mathrm{r}^{*2}}\int_0^\infty \frac{E_{d,0}(k)}{k^2}\sin^2 k\tau\,\mathrm{d}k - \frac{1}{M_\mathrm{r}^{*}}\int_0^\infty \frac{E_{Pd,0}(k)}{k}\sin 2k\tau\,\mathrm{d}k, \tag{41}$$

where $E_{\phi,0}(k)$ denotes the initial value of the three-dimensional spectrum $E_\phi(k)$ of the variable ϕ.

We now introduce the concept of *acoustic equilibrium value*. Let ϕ be a stochastic correlation which evolves on the acoustic timescale t_C. The acoustic timescale t_C was shown to be a fast timescale, which is $O(M_\mathrm{t})$ smaller than the large-eddy turbulence timescale t_T. We denote the acoustic equilibrium value of a variable by subscript A. Then the acoustic equilibrium value ϕ_A of the variable ϕ is the asymptotically stationary value that ϕ attains after many acoustic time intervals. Of course, the acoustic equilibrium is a meaningful quantity only if the acoustic truncation of the equations apply for a sufficiently long time, in other words, only if the acoustic timescale t_C is sufficiently smaller than the other timescales in the problem. The other relevant timescales in the problem of homogeneous turbulence are: the turbulent timescale $t_\mathrm{T} = k/\epsilon$, where k is the turbulent kinetic energy and ϵ is the turbulent dissipation rate; and the timescale associated with the mean velocity gradient $t_\mathrm{M} = (\tilde{u}_{i,j}\tilde{u}_{i,j})^{-\frac{1}{2}}$. Since $t_\mathrm{C}/t_\mathrm{T} = O(M_\mathrm{t})$, and in usual shear-driven turbulent flows $t_\mathrm{M}/t_\mathrm{T} = O(1)$, the acoustic equilibrium is formally realizable when $M_\mathrm{t} \ll 1$. It should be noted that the acoustic equilibrium value corresponds to a quasi-equilibrium phase during the evolution of the variable; the variable remains stationary over the time interval $t_\mathrm{C} \ll t \ll t_\mathrm{T}$.

Mathematically, the acoustic equilibrium value to which $\overline{P'^2}$ evolves on the fast timescale t_C is obtained by evaluating (41) in the limit $\tau \to \infty$. The Riemann–Lebesgue theorem, which states that $\int_a^b f(k)\,\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}k \to 0$ as $t \to \infty$ provided $\int_a^b |f(k)|\mathrm{d}k$ exists, is used to help evaluate this limit, and the following expression for the acoustic equilibrium $(\overline{P'^2})_\mathrm{A}$ of the pressure variance is obtained:

$$(\overline{P'^2})_\mathrm{A} = \frac{1}{2}\left[(\overline{P'^2})_0 + \frac{1}{M_\mathrm{r}^{*2}}\int_0^\infty \frac{E_{d,0}(k)}{k^2}\,\mathrm{d}k\right]. \tag{42}$$

Let us denote the compressible portion of q^2 (which is twice the turbulent kinetic energy) by q_C^2; by definition, we have the relation,

$$q_\mathrm{C}^2 = \overline{u_i^{\mathrm{C}'}u_i^{\mathrm{C}'}}.$$

The three-dimensional spectrum $E_d(k)$ of the dilatation and the three-dimensional spectrum $E_{q_\mathrm{C}^2}(k)$ of the compressible portion of q^2 are related by

$$E_d(k) = k^2 E_{q_\mathrm{C}^2}(k). \tag{43}$$

Using this relation, (42), yields

$$(\overline{P'^2})_\mathrm{A} = \tfrac{1}{2}(\overline{P'^2})_0\,[1+F_0], \tag{44}$$

where F_0 denotes the initial value of the non-dimensional parameter F which is defined as,

$$F = \frac{q_\mathrm{C}^2}{M_\mathrm{r}^{*2}\,\overline{P'^2}}. \tag{45}$$

An expression for the acoustic equilibrium of the compressible turbulent kinetic energy $(q_C^2)_A$ is now obtained. After rescaling time by

$$\tau = \frac{t}{M_r},$$

(19) and (20) become

$$\partial_\tau P' + \frac{M_r \gamma}{\delta} u^{C'}_{i,i} = 0, \tag{46}$$

$$\partial_\tau u^{C'}_i + \frac{\delta}{M_r \gamma} P'_{,i} = 0. \tag{47}$$

Multiplying (46) by $(2\delta^2 P')/(\gamma^2 M_r^2)$, multiplying (47) by $2u_i^{C'}$, and adding the two resulting equations gives

$$M_r^{*2} \partial_\tau P'^2 + \partial_\tau u_i^{C'^2} + 2M_r^* [P' u_i^{C'}]_{,i} = 0, \tag{48}$$

where $M_r^* = \delta/(\gamma M_r)$. Averaging (48) gives the following result for homogeneous turbulence,

$$\partial_\tau [q_C^2 + M_r^{*2} \overline{P'^2}] = 0. \tag{49}$$

Thus, the quantity in square brackets in (49), which physically represents the full non-dimensional turbulent energy, does not change on the acoustic timescale, and consequently we have

$$(q_C^2)_A + M_r^{*2}(\overline{P'^2})_A = (q_C^2)_0 + M_r^{*2}(\overline{P'^2})_0. \tag{50}$$

After substituting (44) into (50), we obtain

$$(q_C^2)_A = \tfrac{1}{2}(q_C^2)_0 \left[1 + \frac{1}{F_0}\right], \tag{51}$$

where F is defined by (45).

On dividing (51) by (44), we obtain the interesting result that the acoustic equilibrium of the non-dimensional parameter F is unity;

$$F_A = 1. \tag{52}$$

The physical significance of F is better understood by reverting to dimensional quantities (denoted by superscript *). After some manipulation, F may be written as

$$F = \frac{\rho_r^* q_C^{*2}}{\overline{(p_C^{*\prime})^2}/\gamma p_r^*}. \tag{53}$$

The numerator of (53) is twice the kinetic energy of the compressible component, and the denominator is twice the potential energy of the compressible component. Thus, the result $F_A = 1$ implies that at acoustic equilibrium there is an equipartition between the kinetic and potential components of the compressible energy. Since (52) is a consequence of processes occurring on the acoustic timescale t_C, as long as the other timescales of the problem (such as k/ϵ) are larger than t_C, we have $F \simeq 1$ for later time. Therefore low Mach number, homogeneous, compressible turbulence has an equilibrium structure characterized by

$$F \simeq 1. \tag{54}$$

We note that equipartition is known to be a feature of non-dissipative, linear wave motion. However, it is interesting that there is a component (namely, the compressible mode) in the nonlinear, dissipative turbulent flow which satisfies equipartition for asymptotically low M_t. More important, we will show that direct numerical simulations support the approximate validity of the equipartition result for $M_t < 0.5$, and thus this result applies to turbulence at larger Mach numbers of technical relevance.

We will now derive an alternative expression for F in terms of the turbulent Mach number M_t which will be useful later. On substituting the expression for M_r^* from (33) into (45); recognizing that $\bar{c}$ and $\bar{p}$ are approximately constant on the acoustic time-scale, and are respectively equal to their initial, references values c_r and p_r; we obtain

$$F = \frac{\gamma^2 M_t^2 \chi}{p_c^2}. \tag{55}$$

Here χ denotes the ratio of compressible kinetic energy to the total turbulent kinetic energy, that is,

$$\chi = \frac{q_C^2}{q^2},$$

and p_c is the non-dimensional ratio of the root mean square (r.m.s.) pressure fluctuations to the mean pressure,

$$p_c = \frac{(\overline{p_C'^2})^{\frac{1}{2}}}{\bar{p}}. \tag{56}$$

An expression is now sought for the equilibrium value of the pressure–dilatation $\overline{P'd'}$. Starting with (38) and (39), the pressure–dilatation can be related to the initial conditions of the turbulence through

$$\overline{P'd'} = \int_0^\infty E_{Pd,0}(k)\cos 2k\tau\,\mathrm{d}k - \frac{1}{2M_r^*}\int_0^\infty \frac{E_{d,0}(k)}{k}\sin 2k\tau\,\mathrm{d}k + \tfrac{1}{2}M_r^*\int_0^\infty kE_{P,0}(k)\sin 2k\tau\,\mathrm{d}k. \tag{57}$$

Again using the Riemann–Lebesgue theorem to evaluate the right-hand side of (57) in the limit $\tau \rightarrow \infty$, gives $(\overline{P'd'})_A = 0$.

The acoustic equilibrium value of the dilatational variance $\overline{d'^2}$ is obtained in a similar manner. The expression for $\overline{d'^2}$ is

$$\overline{d'^2}(\tau) = \int_0^\infty E_{d,0}(k)\cos^2 k\tau\,\mathrm{d}k + M_r^{*2}\int_0^\infty k^2 E_{P,0}(k)\sin^2 k\tau\,\mathrm{d}k + M_r^*\int_0^\infty kE_{Pd,0}(k)\sin 2k\tau\,\mathrm{d}k,$$

while the expression for $(\overline{d'^2})_A$ is

$$(\overline{d'^2})_A = \tfrac{1}{2}(\overline{d'^2})_0 + \tfrac{1}{2}M_r^{*2}\int_0^\infty k^2 E_{P,0}(k)\,\mathrm{d}k. \tag{58}$$

Thus, of the two dilatational correlations $\overline{P'd'}$ and $\overline{d'^2}$, the acoustic equilibrium of $\overline{P'd'}$ is zero, while the acoustic equilibrium of the positive definite quantity $\overline{d'^2}$ is nonzero.

We had earlier asserted that, in moderate Mach number, homogeneous turbulence, the influence of compressibility on the vorticity field is much smaller than its influence on the dilatational field. We will now provide theoretical reasons for this result, which supplement the justification provided by the direct simulations of figure 1. It was shown previously that under the acoustic truncation, the compressible velocity fluctuation $u_i^{C'}$ remains irrotational. Thus, since the acoustic truncation is too severe for the purpose of obtaining the leading-order effect of M_t on the vorticity field, the momentum equation for $u_i^{C'}$ has to be reconsidered. The equation for $u_i^{C'}$ is

$$\partial_t u_i^{C'} + u_j^{I'} u_{i,j}^{C'} + u_j^{C'} u_{i,j}^{I'} + u_j^{C'} u_{i,j}^{C'} + \frac{\delta}{\gamma M_r^2} P'_{,i} = 0, \tag{59}$$

where the viscous term has been neglected for simplicity. The equation for the vorticity $\omega_i^{C'}$ acquired by $u_i^{C'}$ is obtained by taking the curl of (59), and is given below

$$\partial_t \omega_i^{C'} - \omega_j^{I'} u_{i,j}^{C'} - \omega_j^{C'} u_{i,j}^{I'} + u_j^{I'} \omega_{i,j}^{C'} + u_j^{C'} \omega_{i,j}^{I'} + \omega_i^{I'} u_{j,j}^{C'} + \omega_i^{C'} u_{j,j}^{C'} + u_j^{C'} \omega_{i,j}^{C'} - \omega_j^{C'} u_{i,j}^{C'} = 0. \tag{60}$$

In deriving (60) we have used the vector identities

$$(\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} = \nabla(\boldsymbol{A}\cdot\boldsymbol{B}) - \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\times(\nabla\times\boldsymbol{B}),$$
$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} - \boldsymbol{B}(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}).$$

Since initially $\omega_i^{C'} = 0$, the dominant terms in (60) after a short time are

$$\partial_t \omega_i^{C'} - \omega_j^{I'} u_{i,j}^{C'} + u_j^{C'} \omega_{i,j}^{I'} + \omega_i^{I'} u_{j,j}^{C'} = 0. \tag{61}$$

Thus, the nonlinear convective term in the Navier–Stokes equations leads to the forcing of the compressible field by the solenoidal, incompressible field and consequent generation of vorticity $\omega_i^{C'}$. The following equation for the enstrophy associated with u_i^C is obtained by multiplying (61) by $\omega_i^{C'}$ and averaging,

$$\tfrac{1}{2}\partial_t \overline{\omega_i^{C'}\omega_i^{C'}} - \overline{\omega_i^{C'}\omega_j^{I'} u_{i,j}^{C'}} + \overline{\omega_i^{C'} u_j^{C'} \omega_{i,j}^{I'}} + \overline{\omega_i^{C'}\omega_i^{I'} u_{j,j}^{C'}} = 0. \tag{62}$$

A simple order of magnitude analysis of the above equation, after recognizing that the correlations between the incompressible and compressible variables must be prorated by the ratio of timescales of $u_i^{C'}$ and $u_i^{I'}$ which is $O(M_t)$, gives the following result

$$\overline{\omega_i^{C'}\omega_i^{C'}} = O(M_t^2)\,\overline{d'^2}. \tag{63}$$

Recalling that the incompressible velocity field $u_i^{I'}$ satisfies the problem (14)–(15), it is clear that the change in the vorticity field induced by compressibility is completely represented by $\omega_i^{C'}$. Finally, from (63), we conclude that the effect of compressibility on the enstrophy is a factor of M_t^2 smaller than its effect on the dilatational variance, and therefore in moderate Mach number, homogeneous turbulence, the solenoidal dissipation $\epsilon_s = \bar{\nu}\overline{\omega'_i\omega'_i}$ is relatively insensitive to M_t in comparison with the compressible dissipation $\epsilon_c = \frac{4}{3}\bar{\nu}\overline{d'^2}$.

To summarize, in this section we have identified certain variables of compressible, homogeneous turbulence which evolve from arbitrary initial conditions on a fast timescale t_C, where $t_C = O(M_t k/\epsilon)$. The time evolution of these variables has a quasi-equilibrium phase in which the variable has a stationary value which we call the acoustic equilibrium value. We have also shown that these variables can be combined into a non-dimensional parameter F which, after starting from an arbitrary initial value, maintains a value of approximately unity; thus

$$F = \frac{\gamma^2 M_t^2 \chi}{p_c^2} \simeq 1, \tag{64}$$

where $M_t = q/\bar{c}$ is the turbulent Mach number, χ is the ratio of compressible turbulent kinetic energy to the full turbulent kinetic energy, γ is the ratio of specific heats, and $p_c = (\overline{p^{C'^2}})^{\frac{1}{2}}/\bar{p}$ is the ratio of the r.m.s. compressible pressure to the mean pressure.

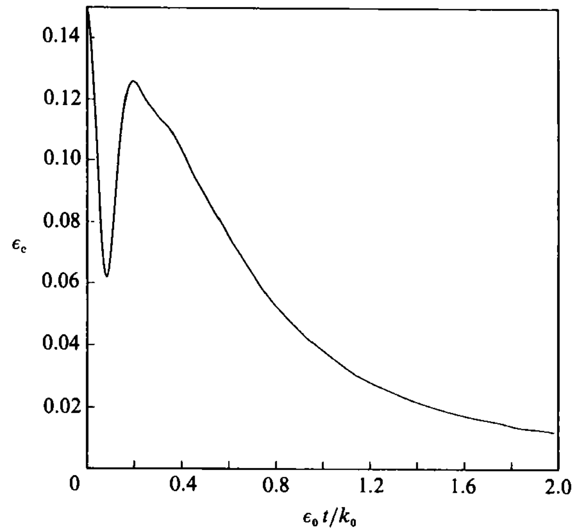

FIGURE 2. Time evolution of the compressible dissipation in a DNS case.

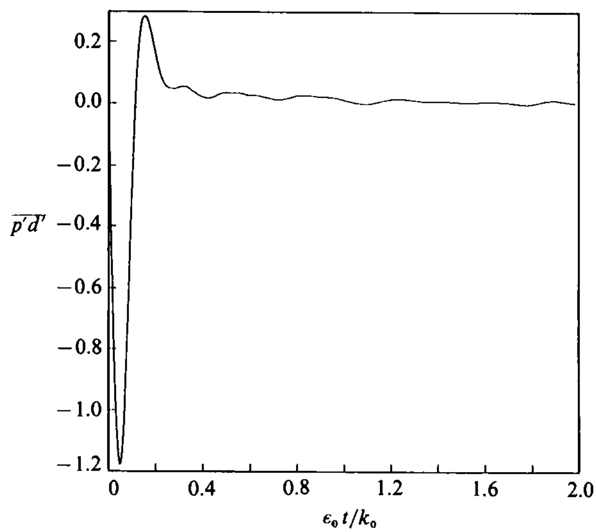

FIGURE 3. Time evolution of the pressure–dilatation for the DNS case of figure 1.

4. Direct numerical simulation of compressible, isotropic turbulence

Three-dimensional direct-numerical simulations (DNS) of compressible, isotropic turbulence were performed on a 64^3 grid for a variety of initial conditions. Details of the algorithm and numerics are provided in Erlebacher *et al.* (1987). The simulations correspond to a nominal turbulence Reynolds number Re_λ (based on the Taylor microscale) of 15. The turbulence Reynolds number is defined by $Re_\lambda = q\lambda/\nu$ where the Taylor microscale $\lambda = (q^2/\overline{\omega_i' \omega_i'})^{\frac{1}{2}}$.

The behaviour of the compressible dissipation and the pressure–dilatation for a case with initial turbulent Mach number $M_{t,0} = 0.5$ is illustrated in figures 2 and 3. The compressible dissipation reaches its acoustic equilibrium value after an initial, fast transient, and then decays with a small superimposed acoustic modulation. The

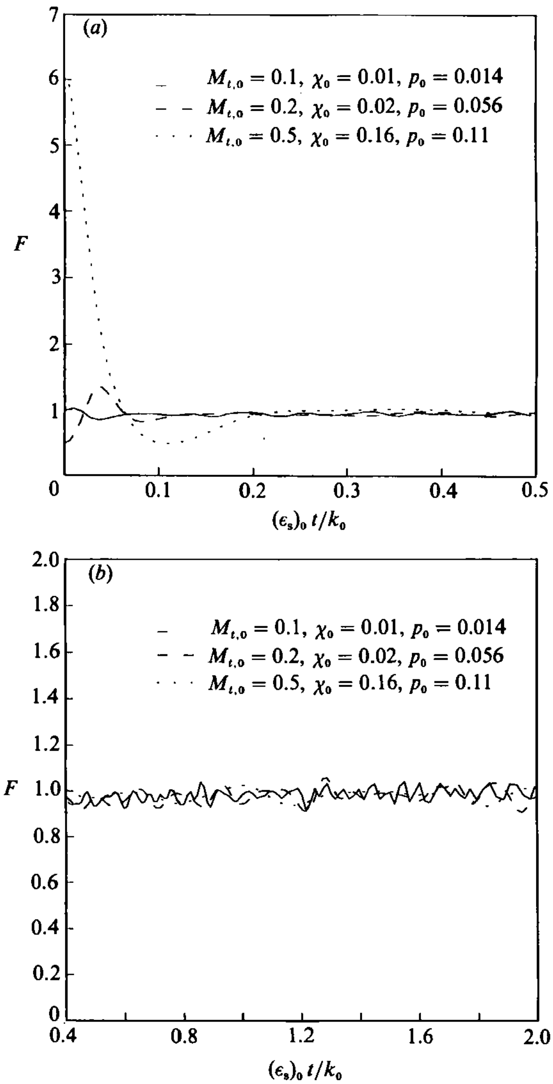

FIGURE 4. (*a*) Early-time history of F for various DNS cases. (*b*) Late-time history of F for various DNS cases.

pressure–dilatation, which can be of either sign, shows a significant acoustic modulation, and tends to be more positive than negative. The direct simulations indicate that in the case of decaying isotropic turbulence, the pressure–dilatation, when averaged over the acoustic oscillations, is positive and smaller than the compressible dissipation.

The asymptotic analysis of the previous section predicted that the non-dimensional parameter $F = (\gamma^2 M_t^2 \chi)/p_c^2$ should be approximately equal to 1. The DNS show that, after starting from a variety of initial values, F indeed reaches a value of unity. Figure 4(*a*) shows the early-time behaviour of F for three representative cases; F attains its acoustic equilibrium value of unity in a non-dimensional time of $O(M_t)$. Figure 4(*b*), which depicts the late-time behaviour of F, shows that F exhibits relatively small excursions from its acoustic equilibrium value of unity. Even though the individual quantities such as M_t^2 decrease by about a factor of 3 in the

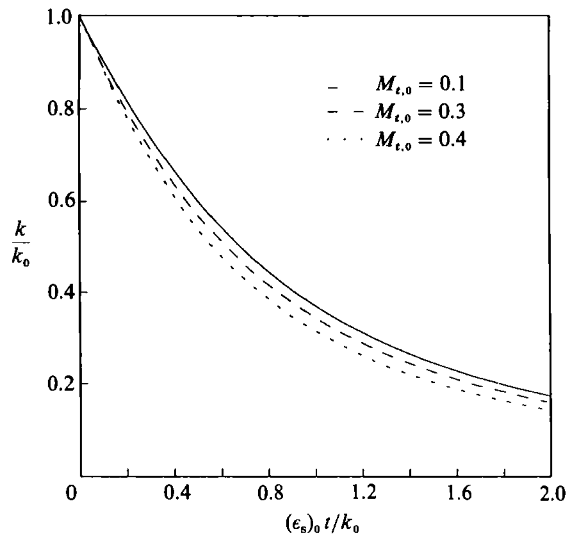

FIGURE 5. DNS results on the decay of isotropic turbulence for various initial conditions.

time interval $0.4 < (\epsilon_s)_0 t/k_0 < 2.0$, the quantity F deviates from its theoretically predicted value of unity by less than 10%. If F is averaged over a few of its oscillations, the deviation of this averaged value from unity would be much smaller than 10%.

Figure 5 shows the DNS results on the decay of the turbulence kinetic energy k for three values of the initial turbulent Mach number $M_{t,0}$. The initial value of the non-dimensional r.m.s. pressure fluctuation was chosen as $M_{t,0}^2$, and, in order to eliminate the initial transient, F was set equal to unity. It is clear that an increase in the compressibility level tends to increase the decay rate of k. Evidently, compressibility leads to an additional source of dissipation for the turbulent kinetic energy.

5. Modelling of the dilatational terms

We will now develop models for the two dilatational terms – the pressure–dilatation, and the compressible dissipation – that appear in the Reynolds stress transport equations. The theoretical analysis indicated that low Mach number homogeneous turbulence is characterized by the relation $F \sim 1$, and the DNS showed that $F \simeq 1$ for turbulent Mach numbers at least up to $M_t = 0.5$ (which, in free shear flows, corresponds to the mean Mach number M being as large as 10). We will now make use of the result $F \simeq 1$ for developing a simple algebraic model of the compressible dissipation ϵ_c. The model essentially relates the turbulent Mach number, which is perhaps the most important quantity characterizing the intrinsic compressibility of high-speed turbulence, to the compressible dissipation.

The compressible fraction of the dissipation rate $\chi_\epsilon = \epsilon_c/\epsilon$ satisfies the following equation:

$$\chi_\epsilon = \frac{\frac{4}{3}\overline{d'^2}}{\overline{\omega_i' \omega_i'} + \frac{4}{3}\overline{d'^2}} = \frac{\chi}{\chi + \frac{3}{4}(\lambda_c/\lambda_s)^2(1-\chi)}, \tag{65}$$

where the compressible Taylor microscale λ_c, the solenoidal Taylor microscale λ_s, and the compressible fraction of turbulent kinetic energy χ, are defined as follows,

$$\lambda_c = (q_C^2/\overline{d'^2})^{\frac{1}{2}}, \tag{66}$$

$$\lambda_s = (q_S^2/\overline{\omega_i' \omega_i'})^{\frac{1}{2}}, \tag{67}$$

$$\chi = q_C^2/q^2. \tag{68}$$

Using (65) we obtain the following expression for f_ϵ, the ratio of compressible dissipation to the solenoidal dissipation:

$$f_\epsilon = \frac{\epsilon_c}{\epsilon_s} = \frac{4\lambda_s^2 \chi}{3\lambda_c^2(1-\chi)}. \tag{69}$$

We assume that for compressible turbulence, $\lambda_c/\lambda_s = O(1)$, and from (69) obtain the asymptotic representation for small χ:

$$f_\epsilon = \beta_1 \chi + O(\chi^2), \tag{70}$$

where $\beta_1 = O(1)$. On using (64), and recognizing that $p_c = O(M_t^2)$, we obtain the following expression from (70):

$$\epsilon_c = \epsilon_s[\alpha_1 M_t^2 + O(M_t^4)], \tag{71}$$

where $\alpha_1 = O(1)$. We now propose the following algebraic model for ϵ_c, which is motivated by (71):

$$\epsilon_c = \alpha_1 M_t^2 \epsilon_s, \tag{72}$$

where α_1 is a constant of $O(1)$, whose numerical value remains to be evaluated. We note that a natural extension of the model (72) for large M_t is to add a term proportional to M_t^4 in (72). For now, we will limit ourselves to the simpler model (72). The solenoidal dissipation rate ϵ_s is calculated using the standard form of the incompressible dissipation rate transport equation.

We also need a model for the pressure–dilatation $\overline{p'd'}$. The asymptotic analysis predicts that the acoustic equilibrium of the compressible pressure–dilatation $\overline{P'd'}$ is zero, while that of the compressible dissipation ϵ_c is non-zero. The DNS indicate that in the case of isotropic turbulence, except for the initial transient, the average of the pressure–dilatation $\overline{p'd'}$ over its oscillations is significantly smaller than the compressible dissipation ϵ_c. Therefore, for the purpose of turbulence modelling, we absorb the effect of $\overline{p'd'}$ in the model of ϵ_c. It should be noted that $\overline{p'd'}$ may be a substantial quantity in flows other than decaying isotropic turbulence, or if eddy shocklets are present.

The model constant α_1 is evaluated by considering the compressible, iso-decay problem. After introducing the models for the compressible dissipation and the pressure–dilatation, and using the standard dissipation transport equation, the governing equations become

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0, \tag{73}$$

$$\frac{\mathrm{d}\tilde{T}}{\mathrm{d}t} = \frac{2}{C_v}\epsilon_s(1+\alpha_1 M_t^2), \tag{74}$$

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\epsilon_s(1+\alpha_1 M_t^2), \tag{75}$$

$$\frac{\mathrm{d}\epsilon_s}{\mathrm{d}t} = -C_{\epsilon 2}\frac{\epsilon_s^2}{k}. \tag{76}$$

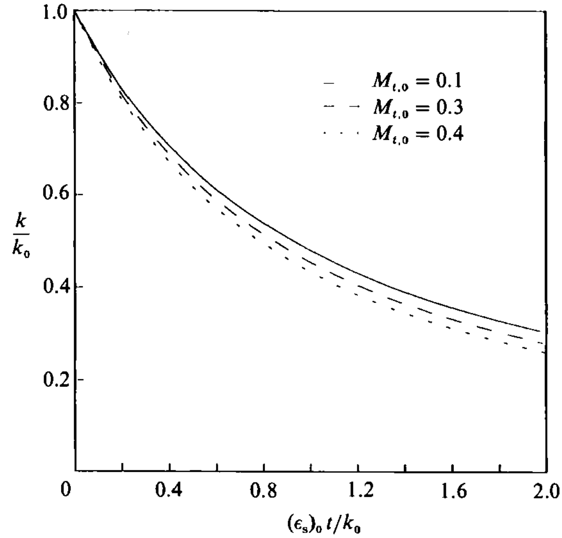

FIGURE 6. Computations of the decay of isotropic turbulence with the model for compressible dissipation ($\alpha_1 = 1$).

Since $M_t^2 = q^2/(\bar{c})^2$, (74) and (75) can be combined into the following equation for M_t^2:

$$\frac{\mathrm{d}(M_t^2)}{\mathrm{d}t} = -\frac{\epsilon_s^2}{k} M_t^2(1+\alpha_1 M_t^2)\,[1+0.5\gamma(\gamma-1)M_t^2]. \tag{77}$$

The term, $-\epsilon_c + \overline{p'd'}$, which is the extra compressible term on the right-hand side of the exact turbulent kinetic energy equation, is replaced by the model, $-\alpha_1 M_t^2 \epsilon_s$, in (75).

Equations (75), (76) and (77) were integrated with a fourth-order, Runge–Kutta scheme using various values for both the model constant α_1 and the initial Mach number $M_{t,0}$. The results of these computations were then compared with DNS. The model coefficient $C_{\epsilon 2}$ was chosen to be 1.83 so as to reproduce the observed decay rate in physical experiments on high-Reynolds-number incompressible turbulence. Since the Reynolds number of the simulations is somewhat low, the turbulence decays faster in the simulations relative to the high-Reynolds-number experiments. Therefore, when comparing model results with the DNS, rather than looking for agreement between the actual value of the decay rate obtained with the model and that obtained with the DNS, we look for agreement regarding the effect of compressibility on the turbulence decay rate. After comparing figure 5 and figure 6 it is clear that, as far as the influence of the turbulent Mach number on the decay rate of the turbulent kinetic energy is concerned, the choice of $\alpha_1 = 1$ gives good agreement between the results of the model and the DNS. Thus, the model for the compressible dissipation becomes

$$\epsilon_c = \alpha_1 M_t^2 \epsilon_s, \tag{78}$$

where the model constant $\alpha_1 = 1$.

The model (78) for the compressible dissipation has been applied by Sarkar & Balakrishnan (1990) to the compressible shear layer, within the framework of a Favre-averaged Reynolds stress closure. Details regarding the other modelling assumptions in the closure and the numerical implementation of the second-order

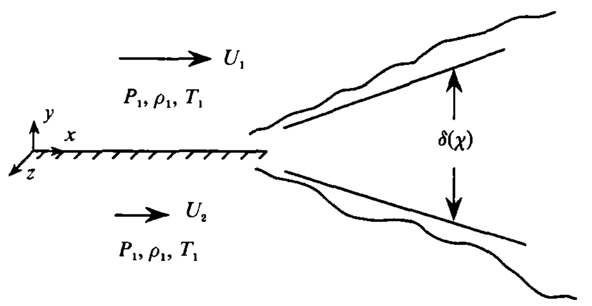

FIGURE 7. Schematic of the compressible shear layer.

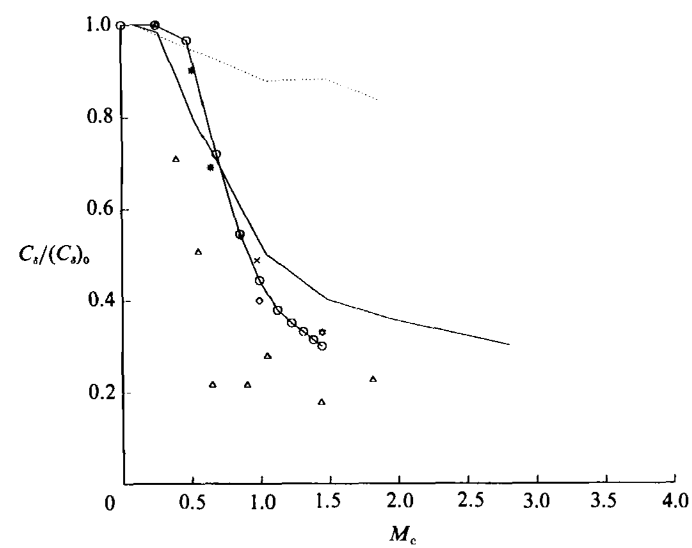

FIGURE 8. Application of the model for compressible dissipation to the compressible shear layer; from Sarkar & Balakrishnan (1990). ——, With compressible dissipation model; ······, without compressible dissipation model; △, Papamoschou & Roshko; –○–, 'Langley experimental curve'; *, Elliott & Samimy; ×, Petrie, Samimy & Addy; ◇, Ikawa & Kubota; ●, Wagner.

closure, and results for various configurations of the compressible shear layer are provided by Sarkar & Balakrishnan. Figure 7 is a schematic of the particular configuration of the shear layer, a few of whose results are given here. A high-speed stream with velocity U_1 mixes with another stream with lower velocity U_2. The free-stream values of the pressure, density and temperature are equal in the two streams. The normalized spreading rate C_δ defined by

$$C_\delta = \frac{\mathrm{d}\delta}{\mathrm{d}x}\left(\frac{U_1+U_2}{U_1-U_2}\right),$$

is the primary variable of interest. The shear-layer thickness $\delta(x)$ is defined to be the distance between the two points of the mean velocity profile where the mean velocity is respectively $U_2+0.1(U_1-U_2)$ and $U_2+0.9(U_1-U_2)$.

Figure 8 shows model predictions and experimental data on the influence of mean compressibility on the spreading rate of the mixing layer. The mean compressibility of the compressible shear layer is characterized by the convective Mach number $M_c = (U_1 - U_2)/(c_1 + c_2)$, where c_1 and c_2 denote the free-stream speed of sound in the two incident streams. In figure 8, we plot the non-dimensional spreading rate $C_\delta/(C_\delta)_0$, where C_δ is the spreading rate of the mixing layer and $(C_\delta)_0$ is the spreading rate of the incompressible mixing layer. Though there is a systematic difference between the data of Papamoschou & Roshko (1988) and the data of the Langley curve (see Kline, Cantwell & Lilley, 1982), which is a consensus representation of various experimental investigations, it is clear that first, the spreading rate decreases significantly when the convective Mach number increases; and second, after the initial decrease, the spreading rate is relatively insensitive to further increases in the convective Mach number. The prediction of the second-order closure, with the model for the compressible dissipation included, is in agreement with both the aforementioned trends exhibited by the experimental data. However, excluding the model of the compressible dissipation from the second-order closure leads to only a mild decrease of spreading rate with increasing Mach number.

6. Conclusions

Asymptotic analysis of the Navier–Stokes equations for high-speed turbulence has isolated a compressible component, which evolves on a fast timescale relative to the incompressible component, and the identification of a non-dimensional parameter F which characterizes a quasi-equilibrium of the compressible component. The variable F evolves from arbitrary initial values on a non-dimensional timescale of $O(M_t)$, attains an equilibrium value of unity, and remains approximately equal to unity for later time. The result $F \simeq 1$, which is formally valid only for turbulent Mach number $M_t \ll 1$, has been shown to hold in the direct numerical simulations (DNS) of isotropic turbulence where M_t was varied between 0.01 and 0.5.

It was established that there is another dilatational correlation – the compressible dissipation – which needs to be modelled in addition to the well-known pressure–dilatation. Both the theoretical analysis and the direct simulations suggest that the compressible dissipation is larger than the pressure–dilatation in low Mach number, homogeneous turbulence. A simple algebraic model, which is based on asymptotic analysis and DNS, has been proposed for the compressible dissipation. The model, which has been applied to the calculation of a high-speed shear layer, was able to capture the dramatically reduced growth rate of the high-speed shear layer.

The present turbulence closure, where dilatational effects are included through a simple model having an algebraic dependence on the turbulent Mach number, will be extended in the future to include transport equations for the thermodynamic turbulence statistics such as the density variance. The consequence of higher-order extensions of the asymptotic theory to compressible turbulence modelling will also be explored.

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